Eur. Phys. J. C 51, 741–752 (2007)
DOI 10.1140/epjc/s10052-007-0311-x

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Nonlinear properties of vielbein massive gravity

S. Groot Nibbelink[1,2,a], M. Peloso[3,b], M. Sexton[3,c]

[1] Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16 und 19, 69120 Heidelberg, Germany
[2] Center for Mathematics and Theoretical Physics, Shanghai Institute for Advanced Study, University of Science and Technology of China, 99 Xiupu Rd, Shanghai 201315, P.R. China
[3] School of Physics and Astronomy, University of Minnesota, 116 Church Street S.E., Minneapolis, MN 55455, USA

Received: 6 February 2007 /
Published online: 3 May 2007 –

Abstract. We propose a nonlinear extension of the Fierz–Pauli mass for the graviton through a functional of the vielbein and an external Minkowski background. The functional generalizes the notion of the measure, since it reduces to a cosmological constant if the external background is formally sent to zero. Such a term and the explicit external background emerge dynamically from a bi-gravity theory, having both a massless and a massive graviton in its spectrum, in a specific limit in which the massless mode decouples, while the massive one couples universally to matter. We investigate the massive theory using the Stückelberg method and providing a 't Hooft–Feynman gauge fixing, in which the tensor, vector and scalar Stückelberg fields decouple. We show that this model has the softest possible ultraviolet behavior that can be expected from any generic (Lorentz-invariant) theory of massive gravity, namely that it becomes strong only at the scale $\Lambda_3 = (m_{\rm g}^2 M_{\rm P})^{1/3}$.

1 Introduction and discussion

Motivated by the observed accelerated expansion of the universe [1–3], and by the theoretical difficulties in ascribing it to a cosmological constant, there has been considerable activity in modifications of gravity at large scales in the past years. For instance, an accelerated expansion can be achieved in bi-gravity models [4], in models in which the Lorentz symmetry is broken by the gradient of a field [5], or in four dimensional models embedded in extra dimensions, as the self-accelerating DGP branch [6–10]. Some of these proposals have properties similar to massive gravity, which is probably the most straightforward and best studied modification of general relativity.

At the linearized level, massive gravity is obtained by adding to the Einstein–Hilbert action a mass term for the metric perturbations $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. The quadratic Lagrangian for this massive spin-two tensor field is given by [11]

$$L = \frac{1}{4} M_{\rm P}^2 \Big\{ h^{\mu\nu} \Big(h_{,\mu\nu} + \Box h_{\mu\nu} - h^{\alpha}_{\mu,\alpha\nu} - h^{\alpha}_{\nu,\alpha\mu} + \eta_{\mu\nu} h^{\alpha\beta}_{\alpha\beta,} - \eta_{\mu\nu}\Box h \Big) + m_{\rm g}^2 \left(h^2 - h^{\mu\nu} h_{\mu\nu} \right) \Big\} , \qquad (1)$$

where $h = \eta^{\mu\nu} h_{\mu\nu}$ is the trace of the metric perturbation. There is a very stringent experimental bound on the graviton mass: $m_{\rm g} \le 7 \times 10^{-41}$ GeV [12], which is close to the inverse of the size of the observable universe. As already observed by Fierz and Pauli (FP) [11], the relative sign between the two mass terms is fixed uniquely by the requirement of having a ghost-free Lorentz-invariant (linear) theory.[1] The massless linear theory $m_{\rm g} = 0$ can be uniquely extended beyond quadratic order using the requirement of general covariance leading to the familiar Einstein–Hilbert action. But because the mass term breaks covariance, it has no unique nonlinear extension.

Covariance can be restored by introducing additional degrees of freedom, as for instance it is done with the Stückelberg method [16]. Another approach is to introduce a second metric into the theory [4, 17, 18]. When one of the two metrics obtains a background expectation value, a mass term for the other metric is generated. Even though such bi-gravity theories are covariant, their completion of the Fierz–Pauli mass term is far from unique, because one can write down an infinite set of invariant nonlinear interactions between the two metrics. It is possible to obtain more uniquely defined bi-gravity theories: [18] considers a bi-gravity model described in terms of the vielbeins (tetrads), rather than metrics. Besides Einstein–Hilbert actions for both sectors it includes all possible cosmological constant-like terms that can be written down using these

[a] e-mail: grootnib@thphys.uni-heidelberg.de
[b] e-mail: peloso@physics.umn.edu
[c] e-mail: sexton@physics.umn.edu

[1] A richer structure of ghost-free mass terms is possible if one is willing to give up Lorentz invariance [13–15].

two vielbeins. The model [18], reviewed in Appendix A, has two spin-two fields, one of which is massless, while the other has a mass term of the FP form (1). Interestingly, the model admits a limit in which the massless mode decouples, while the massive one couples universally to matter.

The goal of the present paper is to investigate the resulting model of massive gravity. We would like to perform an analysis beyond the linearized level and to compare our results with those obtained for generic massive gravity theories. In particular, we want to investigate: when does this massive gravity theory become strong? This question is important, because it is related to the van Dam–Veltman–Zhakarov (vDVZ) discontinuity [19, 20]. The propagator of a massive graviton does not reduce to the massless one in the limit of vanishing graviton mass $m_{\mathrm{g}} \to 0$. The discontinuity between these propagators results in a discontinuity between the perturbative interactions of the massless and massive theories. However, precisely because the interactions of massive gravity become strong, this does not necessarily mean a discontinuity between the final nonperturbative results [16, 21, 22]. Reference [16] showed that the scale at which any nonlinear completion of the FP mass term (1) becomes strong never exceeds $\Lambda_3 = (m_{\mathrm{g}}^2 M_{\mathrm{P}})^{1/3}$. Any generic completion that becomes strong at a smaller energy, can be improved by adding suitable terms to result in a theory that becomes strong at Λ_3. This procedure is in general rather involved. Quite remarkably, we will prove that our model becomes strong precisely at the scale Λ_3, without the need of such additional terms.

The plan of this paper is the following. Section 2 describes the model of massive gravity we want to study in this paper using the vielbein formalism. In Sect. 3, the Stückelberg analysis of [16] is extended to the case of the vielbein. In general, the vielbein formulation of gravity contains more (non-dynamical) fields than the standard metric formulation, which are compensated by additional gauge symmetries (the local Lorentz transformations). We provide the (unique) transformation that fixes these additional degrees of freedom. In Sect. 4 we determine the quadratic action for the graviton and the Stückelberg fields. We present a gauge choice that explicitly decouples the scalar, vector and spin-two degrees of freedom. This choice, which, to our knowledge, has not been provided so far in the literature, leads to particularly simple propagators for the different polarizations. In Sect. 5 we classify the dominant interaction terms, and we show that the scale at which the theory becomes strong is Λ_3. In that section we also compute the tree level amplitude of the $2 \to 2$ scalar scattering, because it can be considered as a typical diagram used to compute the scale at which massive gravity becomes strong. However, we show, by a specific choice of parameters, that this interaction can be made to vanish at this scale, while other interactions still remain strong. These results are summarized in the concluding Sect. 6. In Appendix A we explain how the model of massive gravity studied in this paper can be obtained via a decoupling of the bi-gravity model introduced in [18]. The subsequent appendices are more technical. Appendix B describes some useful properties of a special product that generalizes the definition of the determinant, which is used to write the interaction between the two initial gravitational sectors. The Appendices C and D contain useful intermediate results for the computation of the action in terms of the Stückelberg fields.

2 A massive gravity theory described in the vielbein formalism

In this section we outline the model of massive gravity that will be studied in this paper. The action of this model is

$$S = \frac{1}{2} M_{\mathrm{P}}^2 \int \mathrm{d}^4 x \left\{ \sqrt{-g} R + 3 m_{\mathrm{g}}^2 \left\langle (e-\eta)^2 (e+\eta)^2 \right\rangle \right\} , \tag{2}$$

where we have denoted by $g_{\mu\nu}$ the metric associated to the vielbein $e_{\mu\nu}$. The theory describes a massive graviton and is characterized by the Planck mass M_{P} and the graviton mass m_{g}. To write the cosmological constant-like interaction term we have introduced the notation

$$\langle ABCD \rangle \equiv -\frac{1}{4} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{abcd} A_{\alpha a} B_{\beta b} C_{\gamma c} D_{\delta d} , \tag{3}$$

with ϵ^{abcd} being the totally antisymmetric Levi-Civita tensor. (Some useful properties of this product are collected in Appendix B.) This coupling term is a generalization of the measure, since $\langle A^4 \rangle = |A| = \det(A)$. Using these properties it is not hard to show that the action can be rewritten as

$$S = \frac{1}{2} M_{\mathrm{P}}^2 \int \mathrm{d}^4 x \left\{ \sqrt{-g} \left(R + 3 m_{\mathrm{g}}^2 \right) - \frac{1}{2} m_{\mathrm{g}}^2 \left([e]^2 - [e^2] \right) \right\} , \tag{4}$$

where $[e^n]$ is the trace of $(\eta^{-1} e)^n$. Hence, the final term can be interpreted as a FP mass term for the vielbein. By formally inverting the definition,

$$g_{\mu\nu} = e_{\mu m} \eta^{mn} e_{\nu n} , \tag{5}$$

we could have equivalently expressed it in terms of the metric. However, the resulting expression would appear as a complicated and unmotivated power series of the metric. This model is obtained by considering a specific limit of the bi-gravity theory introduced in [18] as is explained in Appendix A. The specific choice of the action (2) is obtained by imposing the reflection symmetry $e_{\mu m} \to -e_{\mu m}$, and by requiring that the Minkowski background $e_{\mu\nu} = \eta_{\mu\nu}$ is a solution. Indeed, from this last requirement, we see that the second term must factorize the $(e-\eta)^2$ combination, and then the second factor simply follows from the reflection symmetry. This specifies the action (2) uniquely among the several interaction terms that may constructed starting from the vielbein, the Minkowski background, and the product (3).

However, there is still a large arbitrariness in this procedure, given by the choice of the background term. In the action (2) we chose it to be equal to the Minkowski background $\eta_{\mu\nu}$ in Cartesian coordinates. But, already choosing it to be the Minkowski metric in spherical coordinates would result in a different theory, since this term explicitly breaks covariance. We can also obtain more general solutions, starting from a different background metric $e_{\mathrm{b}\,\mu\nu}$ in the action (2), and adding the corresponding source term that would lead to that solution in the standard case since the "mass term" is quadratic in $e_{\mu\nu}-e_{\mathrm{b}\,\mu\nu}$, it would not affect the validity of the solution. However, for definiteness, in the remainder of this paper we will only consider the case where the background is Minkowskian using Cartesian coordinates.

The interaction term in (2) constitutes a particular completion of the FP mass term. Inserting the expression

$$e_{\mu\nu}=\eta_{\mu\nu}+f_{\mu\nu} \tag{6}$$

in the second term of (2) gives

$$\Delta S=\frac{1}{2}m_{\mathrm{g}}^2M_{\mathrm{P}}^2\left\{\left([f]^2-[f^2]\right)+\frac{1}{2}\left([f]^3-3[f][f^2]+2[f^3]\right)\right.\\ \left.+\frac{1}{8}\left([f]^4-6[f]^2[f^2]+3[f^2]^2+8[f][f^3]-6[f^4]\right)\right\}. \tag{7}$$

We stress that this expression is exact (rather than just a perturbative expansion to fourth order). In the remainder of the paper we discuss the effects of the nonlinear interactions in setting the scale at which the model becomes strong.

3 Stückelberg fields

We investigate the graviton interactions in the model outlined in the previous section. Nonlinear interactions of massive gravity can be most easily studied through the Stückelberg formalism, developed in [16]. To do so for our model, we first have to formulate the Stückelberg formalism in terms of the vielbein, rather than the metric (as done in [16]). In this section we describe this computation in some detail.

The Stückelberg formalism consists in performing a series of transformations in (2) and in promoting the parameters of these transformations to new fields. These fields appear in the new action together with additional symmetries, so that the original action can be recovered with a particular gauge choice. However, we can also choose alternative gauges in the new action, where the nonlinear interactions can be computed more easily.

More specifically, we start from a symmetric vielbein perturbation $f_{\mu m}$, and perform a general coordinate transformation $x\to y=y(x)$ combined with a local Lorentz transformation, with $\bar{L}\eta^{-1}\bar{L}^{\mathrm{T}}=\eta$. Performing these transformations into the action (2) results in the replacements

$$\mathrm{d}^4x\to\mathrm{d}^4y=\left|\frac{\partial y}{\partial x}\right|\mathrm{d}^4x\,,\\ e_{\mu a}\to e'_{\mu a}=\frac{\partial x^\alpha}{\partial y^\mu}e_{\alpha b}\eta^{bc}\bar{L}_{ca}\,. \tag{8}$$

We note that, even though $e_{\mu m}$ is assumed to be symmetric, $e'_{\mu m}$ is certainly not automatically symmetric. While the first term of (2) is invariant under these combined transformations, the whole action transforms into the Stückelberg form

$$S\to S_{\mathrm{st}}=\frac{1}{2}M_{\mathrm{P}}^2\int\mathrm{d}^4x\\ \times\left\{\sqrt{-g}R+3m_{\mathrm{g}}^2\left|\frac{\partial y}{\partial x}\right|\left\langle\left(e'-\eta\right)^2\left(e'+\eta\right)^2\right\rangle\right\}, \tag{9}$$

where $e'_{\mu m}$ has been defined in (8). Notice that (8) introduces the Jacobian in front of the cosmological constant term.

The parameters in the local Lorentz transformations are not dynamical; therefore, they can be integrated out using their algebraic equations of motion. Writing

$$\bar{L}_{mn}(x)=L_{mp}(x)\left[\exp(\eta^{-1}b(x))\right]^p{}_n\,, \tag{10}$$

where $b_{\mu\nu}$ is antisymmetric, we expand the action (9) to first order in b:

$$\delta_bS_{\mathrm{st}}=6m_{\mathrm{g}}^2M_{\mathrm{P}}^2\int\mathrm{d}^4x\left|\frac{\partial y}{\partial x}\right|\left\langle e'^3\left(e'\eta^{-1}b\right)-\eta^2e'\left(e'\eta^{-1}b\right)\right\rangle. \tag{11}$$

Using the property (B.1), it is immediate to verify that the first term in (11) vanishes, due to the antisymmetry of b_{mn}. Using (B.3), the remaining term rewrites

$$\delta_bS_{\mathrm{st}}=-\frac{1}{2}m_{\mathrm{g}}^2M_{\mathrm{P}}^2\int\mathrm{d}^4x\left|\frac{\partial y}{\partial x}\right|\\ \times\left\{[e']\left(\eta^{-1}e'\eta^{-1}\right)^{\mu\nu}-\left(\eta^{-1}e'\eta^{-1}e'\eta^{-1}\right)^{\mu\nu}\right\}b_{\mu\nu}\,, \tag{12}$$

where we have used the notation $[e']$ for the trace, see below (7). This contribution vanishes if e' is symmetric, and therefore $\bar{L}=L$ is an on-shell solution. This symmetry of e' together with the requirement that L is a local Lorentz transformation,

$$\left(\frac{\partial x}{\partial y}e\eta^{-1}L\right)^{\mathrm{T}}=\frac{\partial x}{\partial y}e\eta^{-1}L\,,\quad L\eta^{-1}L^{\mathrm{T}}=\eta\,, \tag{13}$$

determines L uniquely in terms of the other fields. See Appendix C for a perturbative construction of L. The freedom in the choice of its sign is fixed by requiring that $L=\mathbb{I}$ if $(\partial x/\partial y)e$ is symmetric.

Therefore, after we solve (13) for L, and we substitute the solution back into (9), we are left with an action that explicitly depends only on the three dynamical fields $f_{\mu\nu}$, a_μ, and ϕ. The latter two fields are obtained by decomposing

$$y^\mu(x)=x^\mu+a^\mu(x)+\partial^\mu\phi(x)\,, \tag{14}$$

into the spin-zero, ϕ, and spin-one, a_μ, polarizations of the massive graviton (this decomposition introduces an additional U(1) gauge symmetry in the theory). These fields are the starting point for the Stückelberg analysis of the nonlinear interactions performed in the next sections.

To summarize the Stückelberg formalism we have employed in this section: in general, there are more degrees of freedom in the vielbein than in the metric, which are compensated by the local Lorentz transformations. This means that in the Stückelberg description additional fields are introduced for both the general coordinate transformations and the local Lorentz transformation. But contrary to the Stückelberg fields associated with the general coordinate transformations, the ones of the local Lorentz transformations are always auxiliary (i.e. non-dynamical) fields. When we enforce their field equations, we ensure that the *composite* vielbein e', defined in (8), remains symmetric *also* after the Stückelberg fields a_μ and ϕ have been introduced. Hence the number of physical degrees of freedom in the vielbein formulation is the same as in the metric formulation discussed in [16].

4 Graviton and Stückelberg propagators

The aim of this section is to compute the propagators of the massive gravity theory defined in Sect. 2 using the Stückelberg field decomposition discussed in the previous section. As usual the propagators can be read off from the quadratic action in the perturbations. However, we will encounter a few complications: first of all, the scalar ϕ obtains a regular kinetic term only after a Weyl rescaling of the graviton field [16]. A second difficulty is that, before inverting the kinetic operator, we need to fix the general coordinate covariance and the additional U(1) gauge symmetry generated by the Stückelberg procedure. Even after the Weyl rescaling, the tensor $f_{\mu\nu}$, the vector a_μ and the scalar ϕ mix with each other at the quadratic level (so that they cannot be used as such to describe independently propagating degrees of freedom). In the following we show how the last two problems can be solved together by choosing 't Hooft–Feynman-like gauge fixing terms.

At quadratic order in $f_{\mu\nu}$, a_μ, ϕ the action (9) is computed using (B.3) and the second order expansion of the matrix L^{-1} given in (C.6). The result takes the rather complicated form

$$
\begin{aligned}
S_2 = \frac{1}{2} M_{\rm P}^2 \int \mathrm{d}^4x \Big\{ & f^{\mu\nu} \Big[f_{,\mu\nu} + \Box f_{\mu\nu} - f^{\alpha}{}_{\mu,\alpha\nu} \\
& \qquad - f^{\alpha}{}_{\nu,\alpha\mu} + \eta_{\mu\nu} \left(f^{\alpha\beta}_{\alpha\beta,} - \Box f \right) \Big] \\
& + m_{\rm g}^2 \Big[f^2 - f^{\mu\nu} f_{\mu\nu} + f^{\mu\nu} \left(\partial_\mu a_\nu + \partial_\nu a_\mu \right) - 2 f \partial^\mu a_\mu \\
& \qquad - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 2 f^{\mu\nu} \partial_\mu \partial_\nu \phi - 2 f \Box \phi \Big] \Big\} \,. \qquad (15)
\end{aligned}
$$

The sign of the kinetic term of the vector a_μ is the standard one (had the sign of the mass term in (2) been opposite, a_μ would have been a ghost). The last two terms on the second line of (15) are the only two in which the scalar ϕ appears, and are not regular kinetic terms for a scalar. Following [16] we perform the linearized Weyl rescaling

$$
f_{\mu\nu} = \hat{f}_{\mu\nu} - \frac{1}{2} m_{\rm g}^2 \eta_{\mu\nu} \phi \qquad (16)
$$

of the graviton, to obtain a regular kinetic term for the scalar ϕ. The quadratic action (15) becomes

$$
\begin{aligned}
S_2 = \frac{1}{2} M_{\rm P}^2 \int \mathrm{d}^4x \Big\{ & \hat{f}^{\mu\nu} \Big[\hat{f}_{,\mu\nu} + \Box \hat{f}_{\mu\nu} - \hat{f}^{\alpha}{}_{\mu,\alpha\nu} \\
& \qquad - \hat{f}^{\alpha}{}_{\nu,\alpha\mu} + \eta_{\mu\nu} \left(\hat{f}^{\alpha\beta}_{\alpha\beta,} - \Box \hat{f} \right) \Big] \\
& + m_{\rm g}^2 \left[\hat{f}^2 - \hat{f}^{\mu\nu} \hat{f}_{\mu\nu} \right] \\
& + m_{\rm g}^2 \Big[\hat{f}^{\mu\nu} \left(\partial_\mu a_\nu + \partial_\nu a_\mu \right) - 2 \hat{f} \partial^\mu a_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
& \qquad + 3 m_{\rm g}^2 \phi \left(\left[\frac{1}{2} \Box + m_{\rm g}^2 \right] \phi + \partial^\mu a_\mu - \hat{f} \right) \Big] \Big\} \,. \qquad (17)
\end{aligned}
$$

Even though we now have obtained a regular kinetic term for the scalar ϕ, this action is still rather complicated and contains quadratic interactions between the three fields $f_{\mu\nu}$, a_μ, and ϕ.

In addition, this action (17) is invariant under the (linearized) general coordinate transformations and a U(1) gauge symmetry

$$
\begin{aligned}
\delta \hat{f}_{\mu\nu} &= \frac{1}{2} \left(\varepsilon_{\mu,\nu} + \varepsilon_{\nu,\mu} \right) - \frac{1}{2} m_{\rm g}^2 \eta_{\mu\nu} \psi \,, \\
\delta a_\mu &= \varepsilon_\mu + \partial_\mu \psi \,, \\
\delta \phi &= -\psi \,. \qquad (18)
\end{aligned}
$$

These gauge symmetries can be fixed by using the following gauge fixing functionals:

$$
\begin{aligned}
\text{GCC}: \quad & \Theta^\mu = \hat{f}^{\mu\nu}{}_{,\mu} - \frac{1}{2} \hat{f}_{,}{}^{\mu} - \frac{1}{2} m_{\rm g}^2 a^\mu \,, \\
\text{U(1)}: \quad & \Theta = a_{\mu,}{}^{\mu} - \hat{f} + 3 m_{\rm g}^2 \phi \,. \qquad (19)
\end{aligned}
$$

Indeed, under the combined gauge transformations (18) these functionals transform as

$$
\delta \Theta^\mu = \frac{1}{2} \left(\Box - m_{\rm g}^2 \right) \varepsilon^\mu \,, \quad \delta \Theta = \left(\Box - m_{\rm g}^2 \right) \psi \,. \qquad (20)
$$

Note that at this order Θ^μ only transforms under general coordinate transformations, while Θ only under the U(1) symmetry. This shows that by suitable gauge transformations these gauge fixing functionals can be made equal to any prescribed functions. When one performs an analysis at the level of equations of motions, it is most convenient to simply set both functionals Θ^μ and Θ to zero. However, since in this section the aim is to obtain simple forms for the propagators, we employ the gauge fixing functionals (19) to define the gauge fixing action

$$
S_{\rm gf} = - M_{\rm P}^2 \int \mathrm{d}^4x \left\{ \Theta^\mu \Theta_\mu + \frac{1}{4} \Theta^2 \right\} \,. \qquad (21)
$$

The use of these gauge fixing functionals can be viewed as a generalization of the 't Hooft R_ξ gauges in spontaneously broken gauge theories. In principle, we can allow for arbitrary normalizations in front of each of the terms in (19): as long as they are suitably chosen, the tensor, vector and scalar modes remain decoupled at the quadratic level. However, in this "Feynman" gauge, the gauge fixed action reduces to the simplest form

$$S_2' = \frac{M_\mathrm{P}^2}{2}\int \mathrm{d}^4x \left\{ \hat{f}^{\mu\nu}\left(\Box - m_\mathrm{g}^2\right)\left(\hat{f}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\hat{f}\right) + \frac{1}{2}m_\mathrm{g}^2 a_\mu \left(\Box - m_\mathrm{g}^2\right) a^\mu + \frac{3}{2}m_\mathrm{g}^4 \phi\left(\Box - m_\mathrm{g}^2\right)\phi \right\} . \tag{22}$$

Hence in the gauge defined by (21) the different spin states $f_{\mu\nu}$, a_μ and ϕ are decoupled, and all have the same mass. In this gauge all components of these fields are dynamical.

This action is very convenient: it is immediate from (22) that the field redefinitions

$$\hat{f}_{\mu\nu} = \frac{1}{M_\mathrm{P}}\hat{f}_{c\mu\nu} , \quad a_\mu = \sqrt{2}\frac{1}{m_\mathrm{g}M_\mathrm{P}}a_{c\mu} , \quad \phi = \sqrt{\frac{2}{3}}\frac{1}{m_\mathrm{g}^2 M_\mathrm{P}}\phi_c , \tag{23}$$

normalizes the fields canonically. Also the propagators for the graviton $\Delta_{\mu\nu\,\rho\sigma}$, the vector field $\Delta_{\mu\,\nu}$ and the scalar Δ become very simple

$$\begin{aligned} \Delta_{\alpha\beta\rho\sigma} &= \frac{1}{2}\left(\eta_{\alpha\rho}\eta_{\beta\sigma} + \eta_{\alpha\sigma}\eta_{\beta\rho} - \eta_{\alpha\beta}\eta_{\rho\sigma}\right)\Delta , \\ \Delta_{\mu\nu} &= \eta_{\mu\nu}\Delta , \\ \Delta &= \frac{\mathrm{i}}{p^2 + m_\mathrm{g}^2} . \end{aligned} \tag{24}$$

These propagators can be obtained from massless propagators in the Feynman gauge by the straightforward substitution: $p^2 \to p^2 + m_\mathrm{g}^2$. Even though the determination of the propagators here was performed in the vielbein formulation, it is obvious that it extends immediately to metric perturbations $h_{\mu\nu}$ as well, since at the linearized level one simply has $h_{\mu\nu} = 2f_{\mu\nu}$. Thus, this computation provides the gauge fixing, which was not explicitly given in [16].

5 Dominant interactions at high energies

After the discussion of the quadratic action and propagators, we now turn to the interactions of this massive gravity theory. The interactions of any massive gravity theory are rather involved, because there are many of them, and they all possess rather complicated tensorial structures. Moreover, the interactions have a polynomial momentum dependence, and become nonperturbative at some "large" energy scale (in this context, "large" is defined with respect to the graviton mass). Therefore it is useful to classify which interactions are the dominant ones at high energies.

As we reviewed in Sect. 3, in the Stückelberg formalism the vector a_μ enters in the "pion" field with one derivative, while the scalar ϕ with two derivatives. Therefore, barring cancellations, the scalar couplings will be in general the largest at high energies. As was noted in [16], the $\phi\phi \to \phi\phi$ scattering generally becomes strong at the scale $\Lambda_5 = (m_\mathrm{g}^4 M_\mathrm{P})^{1/5}$. This scale falls into the regular pattern of scales defined by

$$\Lambda_p = \left(m_\mathrm{g}^{p-1}M_\mathrm{P}\right)^{1/p} , \tag{25}$$

where p can be integer or half integer (notice that Λ_p is a decreasing function of p). All interactions that become strong at scale Λ_3 or below are grouped in Table 1. In particular, the scale Λ_5 appears in $\phi\phi \to \phi\phi$ scattering by combining two three-scalar interactions. It was also shown in [16] that the scale at which this scattering becomes strong can at most be raised to Λ_3. For a generic model of massive gravity, the authors of this reference sketched a procedure to obtain a suitable set of counter terms. We show that the model we are considering automatically satisfies this property.

To give a detailed discussion, we divide the presentation in three subsections. The first one proves that our model does not have any interactions with either only scalars or at most one vector. In the second subsection we show that all tree level processes become strong at a scale that is greater than or equal to Λ_3. In the final subsection we explicitly compute $\phi\phi \to \phi\phi$ scattering, as an example, and we show

Table 1. This table considers all possible interactions that give rise to four-point scatterings that becomes strong at scale Λ_3 or below, using canonically rescaled Stückelberg fields (23). Such scatterings either arise directly from single four-point functions, (**a**), or by combining two three-point functions, (**b**), with a scalar, vector or graviton propagator in between the two vertices

(a) 4 point	scale	(b) 3 point	3 point	scale	3 point	3 point	scale
$(\partial^2\phi)^4$	Λ_4	$(\partial^2\phi)^3$	$(\partial^2\phi)^3$	Λ_5	$(\partial^2\phi)^2(\partial a)$	$(\partial^2\phi)^2(\partial a)$	Λ_4
$(\partial^2\phi)^3(\partial a)$	$\Lambda_{3\frac{1}{2}}$	$(\partial^2\phi)^3$	$(\partial^2\phi)^2(\partial a)$	$\Lambda_{4\frac{1}{2}}$	$(\partial^2\phi)^2(\partial a)$	$(\partial^2\phi)(\partial a)^2$	$\Lambda_{3\frac{1}{2}}$
$(\partial^2\phi)^2(\partial a)^2$	Λ_3	$(\partial^2\phi)^3$	$(\partial^2\phi)(\partial a)^2$	Λ_4	$(\partial^2\phi)^2(\partial a)$	$(\partial^2\phi)^2 f$	$\Lambda_{3\frac{1}{2}}$
$(\partial^2\phi)^3$ f	Λ_3	$(\partial^2\phi)^3$	$(\partial^2\phi)^2 f$	Λ_4	$(\partial^2\phi)^2(\partial a)$	$(\partial a)^3$	Λ_3
		$(\partial^2\phi)^3$	$(\partial a)^3$	$\Lambda_{3\frac{1}{2}}$	$(\partial^2\phi)^2(\partial a)$	$(\partial^2\phi)(\partial a)f$	Λ_3
		$(\partial^2\phi)^3$	$(\partial^2\phi)(\partial a)f$	$\Lambda_{3\frac{1}{2}}$	$(\partial^2\phi)^2 f$	$(\partial^2\phi)^2 f$	Λ_3
		$(\partial^2\phi)^3$	$(\partial a)^2 f$	Λ_3	$(\partial^2\phi)^2 f$	$(\partial^2\phi)(\partial a)^2$	Λ_3
		$(\partial^2\phi)^3$	$(\partial^2\phi)f^2$	Λ_3	$(\partial^2\phi)(\partial a)^2$	$(\partial^2\phi)(\partial a)^2$	Λ_3

that it is possible to extend the model such that this amplitude vanishes altogether.

5.1 Absence of interactions with only scalars or at most one vector

In the next two subsections we want to show that our theory does not become strong below the scale Λ_3. Here we prove that the most dangerous interactions, which have the schematic forms $(\partial^2\phi)^n$ and $(\partial^2\phi)^n(\partial a)$, are all absent in our model. In these subsections we consider the scalar interactions before Weyl rescaling: the additional scalar interactions introduced by the Weyl rescaling will have an additional factor m_g^2 from (16), and they therefore have the same strength as tensor interactions. The interaction term of (9) can be written as

$$\Delta S_\mathrm{st} = \frac{3}{2} m_\mathrm{g}^2 M_\mathrm{P}^2 \int \mathrm{d}^4x \left\langle (e-\Pi)^2 (e+\Pi)^2 \right\rangle , \quad (26)$$

where we have introduced the short-hand notation

$$\Pi_{\mu a} = \frac{\partial y^\nu}{\partial x^\mu} \eta_{\nu b} \left(L^{-1}\right)^{bc} \eta_{ca} , \quad (27)$$

where y^ν is expressed by (14) in terms of a_μ and ϕ. In the following, it is often useful to be able to resort to matrix notation to suppress the indices. Aside from the Minkowski metric η, its perturbation f and the Lorentz transformation L (which are matrices by definition), we encode the derivatives of y into the matrices

$$\pi = \left(\frac{\partial y}{\partial x} - \mathbb{I}\right)\eta = \Phi + \frac{1}{2}\left(A + A^\mathrm{T}\right) - \frac{1}{2}F , \quad (28)$$

where $\Phi_{\mu\nu} = \phi_{,\mu\nu}$, $A_{\mu\nu} = a_{\mu,\nu}$ and $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the U(1) field strength. (For us π is defined strictly by the matrix equation (28); our definition differs slightly from the conventions used in [16].)

Now let us first consider interactions with only scalar fields. To single out from (26) the interactions $(\partial^2\phi)^n$, we can simply replace $\Pi_{\mu\nu} \to \eta_{\mu\nu} + \Phi_{\mu\nu}$ and set both $L_{\mu\nu}$ and $e_{\mu\nu}$ equal to the Minkowski metric $\eta_{\mu\nu}$ (ignoring the vector and tensor contributions). We can take $L_{\mu\nu} = \eta_{\mu\nu}$ because in this case e' is automatically symmetric. This gives

$$\Delta S_\mathrm{st} \supset 3 m_\mathrm{g}^2 M_\mathrm{P}^2 \int \mathrm{d}^4x \left\langle 4\eta^2\Phi^2 + 4\eta\Phi^3 + \Phi^4 \right\rangle , \quad (29)$$

where we have employed the matrix notation defined in (28). All these terms vanish upon partial integration. For example, for the first term we obtain

$$\int \mathrm{d}^4x \left\langle 4\eta^2\Phi^2 \right\rangle = -\int \mathrm{d}^4x \epsilon^{abcd} \epsilon_{ab}{}^{\rho\sigma} \phi_{,c\rho}\phi_{,d\sigma} = \int \mathrm{d}^4x \epsilon^{abcd} \epsilon_{ab}{}^{\rho\sigma} \phi_{,\rho}\phi_{,cd\sigma} = 0 , \quad (30)$$

where we have used the definition (3) and the antisymmetry of the ϵ^{abcd} tensor. Similar arguments also apply to the other two terms. Hence, all the interactions of the form $(\partial^2\phi)^n$ vanish. Equivalently this result can be obtained by going to momentum space and realizing that then all matrices are of the form (B.5), for which the angular bracket vanishes as is proven in Appendix B. As anticipated below (7), this shows that all these pure scalar higher derivative interactions naturally vanish, which was the motivation in [25] to consider these combinations.

Also the interactions of the schematic form $(\partial^2\phi)^n(\partial a)$ vanish. To see this, we employ the matrix notation (28), and we again set $e = \eta$ in the interaction term (26). We first observe that all the terms that are linear in F vanish. This is due to the fact that all the other tensorial structures that would multiply F, namely η or Φ, are symmetric, while F itself is antisymmetric. Therefore, we can set both $f_{\mu\nu}$ and $F_{\mu\nu}$ to zero, without losing the term that we are looking for. Doing so, we have a symmetric pion matrix, $\pi = \Phi + \frac{1}{2}A + \frac{1}{2}A^\mathrm{T}$. Once we insert it in (28), we obtain a symmetric e' vielbein already; therefore $L = \eta$. The expression we are looking for can be found by evaluating (26) for $F_{\mu\nu} = f_{\mu\nu} = 0$, $L_{\mu\nu} = \eta_{\mu\nu}$. This results in an interaction action that is again of the form (29) upon making the replacement $\Phi \to \Phi + \frac{1}{2}A + \frac{1}{2}A^\mathrm{T}$. The final step is to single out from this expression the terms that are linear in the vector field. It is clear that they are of the form $\langle A\Phi^n\eta^{3-n}\rangle$. Using a partial integration procedure similar to that of the terms $\langle\Phi^n\eta^{4-n}\rangle$ presented above (or again using (B.5) in momentum space), it is easy to verify that also such terms vanish.

5.2 All interactions become strong at scale Λ_3 or above

We now show that there are no scattering amplitudes that become strong at a smaller scale than Λ_3. We only analyze the theory at the classical level; in particular, we do not consider loop graphs. We first consider the S-matrix elements that correspond to diagrams that contain a single vertex V_{n_g,n_a,n_ϕ}, with n_g tensor, n_a vector and n_ϕ scalar external legs.[2] Taking into account that at a vertex there is momentum conservation encoded in a single overall four dimensional momentum delta function, which scales as $1/E^4$ (where E is the energy in the scattering), we see that such vertex scales as

$$V_{n_g,n_a,n_\phi} \sim \frac{m_\mathrm{g}^2 M_\mathrm{P}^2}{E^4} \left(\frac{1}{M_\mathrm{P}}\right)^{n_g} \left(\frac{E}{m_\mathrm{g} M_\mathrm{P}}\right)^{n_a} \left(\frac{E^2}{m_\mathrm{g}^2 M_\mathrm{P}}\right)^{n_\phi} . \quad (31)$$

where the E in parentheses arise from derivatives (scalars enter in the Stückelberg formalism with two derivatives, while vectors enter with one), and the denominators arise from the canonical normalization. To understand at which energy E such a process becomes strong we need to compute the scaling of the corresponding S-matrix element S_{n_g,n_a,n_ϕ}, in which we integrate over all possible external

[2] We would like to remind the reader that, for reasons explained in the previous subsection, we count the scalar and tensor legs before Weyl rescaling.

momenta. Because the external particles are all on-shell, we integrate over

$$\int \mathrm{d}^4 p_i \delta(p_i^2 - m_i^2) = \int \frac{\mathrm{d}^3 p_i}{2E_{ip_i}} \sim E^2 \,, \tag{32}$$

where in the last step we used that in the high energy regime we are considering $E \gg m_{\mathrm{g}}$. Hence the corresponding S-matrix element scales like

$$S^{1/2}_{n_g,n_a,n_\phi} \sim \left(\frac{m_{\mathrm{g}} M_{\mathrm{P}}}{E^2}\right)^2 \left(\frac{E}{M_{\mathrm{P}}}\right)^{n_g} \times \left(\frac{E^2}{m_{\mathrm{g}} M_{\mathrm{P}}}\right)^{n_a} \left(\frac{E^3}{m_{\mathrm{g}}^2 M_{\mathrm{P}}}\right)^{n_\phi} . \tag{33}$$

(For notational convenience we have taken the square root of the S-matrix element; the interaction becomes strong when the matrix element, or its square root, exceed one.)

Now we analyze the strength of general interactions with a single vertex. Because we are interested in scatterings, the total number of external legs $n_g + n_a + n_\phi \geq 3$. In total we can distinguish five different cases. (i) $n_a = n_g = 0$; (ii) $n_a = n_\phi = 0$; (iii) $n_g, n_\phi \geq 1$; (iv) $n_a = 1$, and finally (v) $n_a \geq 2$. We have shown in the previous subsection that the model does not have vertices with only scalars, hence there are no processes that correspond to the first case, (i). The second case, (ii), involves interactions among only tensors. The corresponding S-matrix elements

$$S^{1/2}_{n_g,n_a=n_\phi=0} \sim \left(\frac{m_{\mathrm{g}}}{M_{\mathrm{P}}}\right)^2 \left(\frac{E}{M_{\mathrm{P}}}\right)^{n_g-4} \tag{34}$$

are bounded from above for $n_g \leq 4$ and therefore do not lead to strong coupling behavior. For $n_g \geq 5$ these interactions become strong at an energy scale that is greater than the Planck scale M_{P}. Next, let us consider interactions of type (iii) that involve at least one scalar and one graviton. By rewriting the general expression (33) as

$$S^{1/2}_{n_g\geq 1,n_a,n_\phi\geq 1} \sim \left(\frac{E}{\Lambda_1}\right)^{n_g-1} \left(\frac{E}{\Lambda_2}\right)^{2n_a} \left(\frac{E}{\Lambda_3}\right)^{3(n_\phi-1)} , \tag{35}$$

we conclude that these interactions are still weak at energy scales lower than Λ_3, because all the exponents of the factors are nonnegative, and the scales $\Lambda_3 < \Lambda_2 < \Lambda_1 = M_{\mathrm{P}}$ are ordered hierarchically; see their definitions (25). Let us turn to interactions of type (iv), with a single vector, $n_a = 1$. From the previous subsection we know that if there are no tensors, $n_g = 0$, the amplitude vanishes. The situation with at least one tensor and one scalar constitutes a special case of (iii). The remaining possibility of type (iv), no scalars and $n_g \geq 2$, becomes strong above the Planck scale M_{P} (analogously to case (ii)). This is clear when we rewrite

$$S^{1/2}_{n_g\geq 2,n_a=1,n_\phi=0} \sim \left(\frac{m_{\mathrm{g}}}{M_{\mathrm{P}}}\right)^2 \left(\frac{E}{M_{\mathrm{P}}}\right)^{n_g-2} . \tag{36}$$

Finally, also case (v) remains weak at all scales lower than Λ_3, because we can write

$$S^{1/2}_{n_g,n_a\geq 2,n_\phi} \sim \left(\frac{E}{\Lambda_1}\right)^{n_g} \left(\frac{E}{\Lambda_2}\right)^{2(n_a-2)} \left(\frac{E}{\Lambda_3}\right)^{3n_\phi} . \tag{37}$$

Hence, we showed that, in this model, all n-leg interactions with a single vertex become strong at the scale Λ_3 or higher.

Finally, let us discuss tree diagrams that contain more than one vertex. Such diagrams can be obtained recursively by combining tree level diagrams with less vertices inside. Whenever we combine two such diagrams, we loose two external lines and hence two factors of E^2 in the S-matrix. At the same time we gain a factor E^2 in the amplitude, since we have an additional momentum integral over a propagator: $\int \mathrm{d}^4 p/p^2 \sim E^2$. (Because we consider tree diagrams all momenta inside a given diagram are fixed by momentum conservation.) Therefore, the scaling of the S-matrix element of the combined diagram is the same as of the original two disconnected diagrams. We can repeat this argument recursively, as we split any tree level diagram in a series of single vertex diagrams. Since we already saw that all single vertex scatterings become strong at least at the scale Λ_3, this argument shows that this is the case for any tree level scatterings.[3]

5.3 Can the $\phi\phi \to \phi\phi$ scattering amplitude vanish?

The interaction vertices given in Sect. 5.1 can be employed to obtain several four-point amplitudes that (if not vanishing) all become strong at the scale Λ_3. These amplitudes correspond to the scattering processes $\phi\phi \to \phi\phi$, $\phi\hat{f} \to \phi\hat{f}$, $a\phi \to a\phi$, $aa \to aa$, and $\phi\hat{f} \to a\ a$, plus the crossed processes. As an example, we compute the $\phi\phi \to \phi\phi$ scattering at tree level. In particular, we want to investigate whether it is possible to have a model where the full leading $\phi\phi \to \phi\phi$ scattering vanishes at tree level.

As will become clear below, the amplitude for $\phi\phi \to \phi\phi$ does not vanish in the model (2). Therefore, we consider the slightly generalized interaction term

$$\Delta S_{\mathrm{gen}} = \frac{6 m_{\mathrm{g}}^2 M_{\mathrm{P}}^2}{4+\alpha+\beta} \int \mathrm{d}^4 x \times \left\langle (e-\eta)^2 \left(e^2 + (2+\alpha)\eta e + (1+\beta)\eta^2\right)\right\rangle , \tag{38}$$

where the real parameters α and β are arbitrary. The normalization factor $4+\alpha+\beta$ in (38) is chosen such that m_{g}^2 still represents the graviton mass. Note that for $\alpha = \beta = 0$

[3] All the arguments in this section ignore any possible interference between different diagrams. Interferences can only soften the scattering amplitudes, so they do not affect the conclusion that the interactions are weak at energies smaller than Λ_3.

we recover the terms in (2), which is symmetric under reflection of the vielbein. (Also in this more general model, the interactions that can potentially become strong at a scale lower than Λ_3 vanish for the same reasons we have discussed in the previous subsections.) To be able to directly compare various scattering amplitudes, we use the canonically normalized fields defined in (23). The leading interactions of the action (38) can be expressed as

$$\begin{aligned}
\Delta L_{\rm gen} \supset &-\frac{1}{\sqrt{6}}\frac{1}{m_{\rm g}^2 M_{\rm P}}\left\{\frac{2+3b}{2}[F_c^2\,\Phi_c]-\frac{1+3b}{4}[F_cF_c]\Box\phi_c\right\}\\
&+\frac{1}{3}\frac{1+3b}{m_{\rm g}^2 M_{\rm P}}\left\{\frac{1}{\sqrt{6}}\phi_c\left((\Box\phi_c)^2-[\Phi_c^2]\right)-\frac{1}{2}[\hat f_c](\Box\phi_c)^2+[\hat f_c\Phi_c^2]\right.\\
&\qquad\left.-\frac{1}{2}[\hat f_c][\Phi_c^2]-[\hat f_c\Phi_c]\Box\phi_c\right\}\\
&+\frac{a-2b}{m_{\rm g}^4 M_{\rm P}^2}\left\{\frac{1}{3\sqrt{6}}\left((\Box\phi_c)^3[\hat f_c]-3\Box\phi_c[\hat f_c][\Phi_c^2]\right.\right.\\
&\qquad-3(\Box\phi_c)^2[\Phi_c\hat f_c]+3[\Phi_c^2][\Phi_c\hat f_c]+2[\hat f_c]\,[\Phi_c^3]\\
&\qquad\left.-6[\hat f_c\Phi_c^3]+6\Box\phi_c[\hat f_c\Phi_c^2]\right)\\
&\qquad+\frac{1}{8}\left([\Phi_c^2][F_c^2]-(\Box\phi_c)^2[F_c]^2\right)\\
&\qquad\left.-\frac{1}{18}\phi_c\left((\Box\phi_c)^3-3\Box\phi_c[\Phi_c^2]+2[\Phi_c^3]\right)\right\}\\
&+\frac{1}{4}\frac{1}{m_{\rm g}^4 M_{\rm P}^2}\left\{\left(\frac{2}{3}+5b-2a\right)[\Phi_c^2F_c^2]\right.\\
&\qquad+\left(\frac{2}{3}+3b-a\right)[\Phi_cF_c\Phi_cF_c]\\
&\qquad\left.-\left(\frac{1}{3}+5b-2a\right)\Box\phi_c[\Phi_cF_c^2]\right\}, \qquad (39)
\end{aligned}$$

where we have defined the parameters

$$a=\frac{\alpha}{4+\alpha+\beta}\,,\qquad b=\frac{\beta}{4+\alpha+\beta}\,. \qquad (40)$$

To obtain these interactions we first performed a Stückelberg transformation to the action (38); we then expanded it up to first order in f and fourth order in π, and we finally performed the linearized Weyl rescaling. To single out only the leading terms, we then substituted $\pi=A+\Phi$ and kept only those terms with the highest power of Φ, since they contain the greatest number of derivatives. In this way, we found the expansion (C.8) given in Appendix C for Π defined in (27). Finally we worked out the brackets $\langle\dots\rangle$ in terms of traces $[\dots]$, using the identities of Appendix B.

This fourth order leading expression is rather involved for generic values of a and b. However, notice that if $a=2b$ the entire term with the third and braces of (39) vanish. In particular, there are no pure scalar interactions anymore. This is also the case for the model (2) with $a=b=0$, which we are mostly concerned with in this paper. However, if in addition $b=-1/3$, also the second braced term and the very last term on the last line vanishes, and only three different interactions survive: one two-scalar vector interaction and two two-scalar two-vector interactions. In particular there are no interactions with tensors left. Hence the only possible four-point scatterings involve two scalars and two vectors.

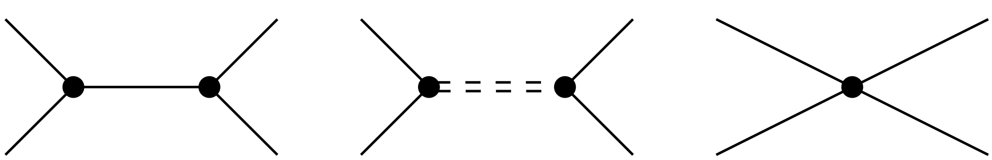
We compute the leading tree level $\phi\phi\to\phi\phi$ scattering, which becomes strong at the scale Λ_3 for generic values of a and b, using the leading expansion (39). The process is described by the three diagrams shown in Fig. 1. The computation of these diagrams is a straightforward exercise in Feynman graph computations and therefore only the result is given here. We neglect the graviton mass against the external and internal momenta of the scattering. In terms of the standard Mandelstam variables s, t and u, the amplitude reads

$$\mathcal{M}(\phi\phi\to\phi\phi)=\left[-\frac{(1+3b)^2}{8}+\frac{(1+3b)^2}{12}+\frac{a-2b}{3}\right]\frac{stu}{m_{\rm g}^4M_{\rm P}^2}\,. \qquad (41)$$

The three terms correspond to the diagrams a, b and c given in Fig. 1, respectively. The total amplitude expressed in terms of the center of mass energy E equals

$$\mathcal{M}(\phi\phi\to\phi\phi)=-\frac{2}{3}\left[8(2b-a)+(1+3\,b)^2\right]\frac{E^6}{M_{\rm P}^2m_{\rm g}^4}\sin^2\theta\,, \qquad (42)$$

where θ defines the angle between the momenta of an ingoing and an outgoing particle in the center of mass frame. We see that, besides the special point $a=2b=-2/3$, there is a whole parabola $a(b)=2b+(1+3b)^2/8$ such that the full tree level $\phi\phi\to\phi\phi$ scattering vanishes at the scale Λ_3. We emphasize that this four-point scalar scattering alone is not sufficient to understand the strong coupling dynamics of massive gravity theories and of our model in particular.



(a) (b) (c)

Fig. 1. This diagram displays the three diagrams that contribute to the four scalar scattering at the scale Λ_3, which result from (39). In the first diagram the scalar ϕ is exchanged, while in the second diagram the graviton $\hat f_{\mu\nu}$ is the mediating particle. The last diagram results from the four-point interactions of scalars

6 Conclusions

The model of massive gravity considered in this paper is standard general relativity with a cosmological constant and a Fierz–Pauli mass term for the vielbein. By requiring a reflection symmetry of the vielbein, $e_{\mu a} \to -e_{\mu a}$, this theory, like standard Fierz–Pauli massive gravity, is described by only two parameters: the Planck scale and the graviton mass. As such it constitutes one of the simplest nonlinear extensions of massive gravity theories. This model can be obtained from a bi-gravity theory with an extremely simple interaction term between the two sectors. The bi-gravity theory contains a massless and a massive graviton state, and it admits a limit in which the massless graviton decouples [18], as we show in Appendix A. In the present work we studied the model of a single massive graviton that emerges in this limit.

We have investigated whether the simplicity of the model has some physical implications, or only has aesthetic merit. To do so, we studied this model at the nonlinear level (since, at the linear level, it is equal to the standard Fierz–Pauli model). As the present model is a special type of massive gravity, it shares the problems that massive gravity theories have in general. The main one is that they are plagued by ghosts at the nonlinear level [23, 24], which can never be avoided at the quartic order in the perturbations [25]. A related difficulty is that the self-interactions of a massive graviton become strong at macroscopic distances from a source [21]. In general, the scattering becomes strong at the energy scale $\Lambda_5 \equiv \left(m_{\rm g}^4 M_{\rm P}\right)^{1/5}$. However, by adding to the starting theory a set of suitable nonlinear interactions, this scale can be raised up to $\Lambda_3 \equiv \left(m_{\rm g}^2 M_{\rm P}\right)^{1/3}$ at most [16]. Even though our theory does not solve these essential problems of massive gravity theory, it provides a minimal model that has self-interactions that become strong at the highest possible scale Λ_3. Therefore, the model here discussed may be considered as a prototype of massive gravity, since, in addition to its minimal formulation, has the best behavior that we can hope to obtain for such theories.

We conclude with a note on the use of the vielbein rather than the metric formulation. In (7), we wrote the (exact) Lagrangian of the model in terms of metric perturbations. Reference [25] already showed that special combinations of terms have the consequence of removing from the theory the self-interactions that involve only the scalar polarization of the graviton, before Weyl rescaling (such interactions – if present – would lower the strong scale below Λ_3). However, what in the metric computation appears only as a computational result (obtained by allowing for arbitrary coefficients, and then finding which combination eliminates the unwanted terms), in the vielbein formulation corresponds to one of the simplest combination of a Fierz–Pauli mass for vielbein perturbations and a cosmological constant. This suggests that, despite that it is very rarely considered, the vielbein approach may be more suitable for the study of massive gravity, and, hopefully, for finding the improvements that it still requires.

Acknowledgements. The work of M.P. and M.S. was partially supported by DOE grant DE-FG02-94ER-40823, and by a grant from the Office of the Dean of the Graduate School at the University of Minnesota.

Appendix A: Derivation from a bi-gravity theory

In this appendix we explain how the massive gravity model discussed in this paper can be derived from the bi-gravity theory discussed in [18]. One starts from the two metrics $\tilde{g}_{+\mu\nu}$ and $\tilde{g}_{-\mu\nu}$, which we rewrite in terms of two vielbeins $\tilde{e}_{+\,\mu m}$ and $\tilde{e}_{-\,\mu m}$, using the standard definition

$$\tilde{g}_{\pm\mu\nu} = \tilde{e}_{\pm\mu m}\eta^{mn}\tilde{e}_{\pm\nu n}\,. \tag{A.1}$$

The action for the model is

$$S_{\rm bi} = \int {\rm d}^4 x \left\{ \sqrt{-\tilde{g}_+}\left(\frac{1}{2}M_+^2\tilde{R}_+ + \Lambda_+\right) + \sqrt{-\tilde{g}_-}\left(\frac{1}{2}M_-^2\tilde{R}_- + \Lambda_-\right) - 2\Lambda_0\left\langle \tilde{e}_+^2\tilde{e}_-^2\right\rangle\right\}. \tag{A.2}$$

The two gravitational sectors are characterized by the two "Planck masses" $M_\pm$ and the cosmological constants Λ_0 and $\Lambda_\pm$.[4] The double covariance of the model is broken by the last term.

Let us now proceed to the study of the spectrum of the model for the Minkowski background. We set

$$e_{\pm\mu\nu} = \eta_{\mu\nu} + f_{\pm\mu\nu}\,, \tag{A.3}$$

and we expand the action (A.2) at the quadratic level in the perturbations $f_{\pm\,\mu\nu}$. The resulting action is not diagonal in terms of these two modes; however, it can be diagonalized through the redefinition

$$\begin{pmatrix} f_{+,\mu\nu} \\ f_{-,\mu\nu}\end{pmatrix} = \frac{1}{r+\frac{1}{r}}\begin{pmatrix} 1 & -\frac{1}{r} \\ 1 & r\end{pmatrix}\begin{pmatrix} f_{0,\mu\nu} \\ f_{\mu\nu}\end{pmatrix}, \quad r = \frac{M_+}{M_-}c^2\,. \tag{A.4}$$

The mode $f_{0\,\mu\nu}$ is massless, while $f_{\mu\nu}$ has a Fierz–Pauli mass $m_{\rm g}$, which is related to the cosmological constant Λ_0 by[5]

$$\Lambda_0 = \frac{3}{2}m_{\rm g}^2 M_{\rm P}^2\,, \quad M_{\rm P}^2 = M_+ M_- \left(r + \frac{1}{r}\right)^{-1}. \tag{A.5}$$

The Planck mass $M_{\rm P}$ (as obtained from the kinetic terms) is found to be identical for both gravitons. The graviton mass $m_{\rm g}$ vanishes for $\Lambda_0 = 0$, as a consequence of the enlarged covariance.

The massless graviton decouples in the limit of either $M_+ \to \infty$ (with finite M_-), or $M_- \to \infty$ (with finite M_+).

[4] As we see below Λ_0 needs to be positive to avoid a tachyonic mass for the graviton.

[5] We correct a typo in the definition of the mass appearing in [18].

Both the Planck and graviton mass appearing in (A.5) are finite in these limits, so that the quadratic actions for the massless and massive modes remain finite. However, all the nonlinear interactions involving the massless mode $f_{0\mu\nu}$ vanish in either limit. For instance, in the first limit, we have at leading order $f_+ \propto f_0/M_+, f$ and $f_- \propto f_0/M_+, f/M_+^2$ (suppressing for shortness the tensorial indices). Therefore, writing the original action (A.2) in terms of f and f_0, we see that any term containing the massless mode is suppressed by a negative power of M_+, and therefore vanishes as $M_+ \to \infty$. The only exception is the quadratic kinetic term from the $+$ sector, which results in a finite kinetic term for the massless graviton with the Planck mass (A.5). In this limit, the massive graviton is identified with f_-, and a theory of a single massive graviton is obtained by coupling all matter fields in the $-$ sector. Obviously, an analogous situation is obtained in the other limit.

To obtain the model of this paper, we study the theory in either of the two (equivalent) limits by ignoring the decoupled massless graviton. We also restrict our attention to the Minkowski background. For instance, in the $M_+ \to \infty$ limit, this gives $\tilde{e}_{+\mu\nu} = c\eta_{\mu\nu}$, $\tilde{e}_{-\mu\nu} = c^{-1}e_{\mu\nu} = c^{-1}(\eta_{\mu\nu} + f_{\mu\nu})$, and the action (A.2) reduces to (2).

Appendix B: Properties of $\langle ABCD\rangle$

In this appendix we collect various helpful properties of the angular bracket $\langle ABCD\rangle$ defined in (3). First of all the ordering of the matrices $A,\ldots,D$ is irrelevant in this expression, because of the two Levi-Civita tensors in its definition. For the same reason, this product is invariant under the simultaneous transposition of all four matrices. As we remarked in the main text, it generalizes the notion of a determinant, in the sense that $\langle A^4\rangle = \det(A) = |A|$. However, $\langle A^2B^2\rangle$ cannot be written as a determinant. The property of a determinant that the determinant of a product of matrices is equal to the product of their determinants generalizes to

$$\langle (aAb)(aBb)(aCb)(aDb)\rangle = |a|\langle ABCD\rangle|b| \,, \quad \text{(B.1)}$$

for any matrices a, b, A, B, C, D.

The angular bracket can be rewritten in terms of traces $[A] = \eta^{ab}A_{ba}$; the resulting expression is rather involved:

$$\begin{aligned}\langle ABCD\rangle = \frac{1}{24}\,&([A][B][C][D] - [A][B][CD] - [A][D][BC]\\ &- [B][D][AC] - [AB][C][D] - [A][BD][C] - [AD][B][C]\\ &+ [AB][CD] + [AC][BD] + [AD][BC] + [A][BCD]\\ &+ [A][BDC] + [B][ADC] + [B][ACD] + [C][ABD]\\ &+ [C][ADB] + [D][ABC] + [D][ACB] - [ABCD]\\ &-[ADBC] - [ACDB] - [ABDC] - [ACBD] - [ADCB])\,.\end{aligned} \quad \text{(B.2)}$$

However, when some of its entries are equal to the Minkowski metric η, its expression simplifies considerably:

$$\begin{aligned}&\langle\eta^4\rangle = 1\,, \quad \langle\eta^3 A\rangle = \frac{1}{4}[A]\,,\\ &\langle\eta^2 A\, B\rangle = \frac{1}{12}\,([A][B] - [AB])\\ &\langle\eta ABC\rangle = \frac{1}{24}\,([A][B][C] - [A][BC] - [B][AC]\\ &\qquad - [C][AB] + [ABC] + [ACB])\,.\end{aligned} \quad \text{(B.3)}$$

Also, we can use (B.2) to express the determinant of a matrix in terms of traces:

$$\begin{aligned}|A| &= \langle A^4\rangle\\ &= \frac{1}{24}\left([A]^4 - 6[A]^2[A^2] + 3[A^2]^2 + 8[A][A^3] - 6[A^4]\right)\,.\end{aligned} \quad \text{(B.4)}$$

Finally, if the matrices $A,\ldots,D$ are formed from five arbitrary vectors p, q, r, s and t,

$$A_{\alpha a} = p_\alpha q_a\,, \quad B_{\beta b} = p_\beta r_b\,, \quad C_{\gamma c} = p_\gamma s_c\,, \quad D_{\delta d} = p_\delta t_d\,, \quad \text{(B.5)}$$

then, by the antisymmetry with respect to the exchange of any two of the $p_\alpha, p_\beta, p_\gamma, p_\delta$ inside the bracket expression, we find $\langle ABCD\rangle = \langle\eta ABC\rangle = \langle\eta^2 AB\rangle = 0$.

Appendix C: Perturbative expansions

This appendix is devoted to some technical details of the perturbative expansions, which we use in the main part of the text to determine the interactions of the massive gravity theory in the Stückelberg formulation. The interactions are encoded in the expression (26), where the matrix Π is defined in (27). The first step for the computation of Π is to determine L from (13). Since the interactions of the graviton are determined by an expansion around the Minkowski background, we consider the infinitesimal general coordinate transformation

$$\mathbb{I} + \epsilon\eta^{-1} = \frac{\partial x}{\partial y}e\eta^{-1}\,, \quad \text{(C.1)}$$

and we expand $L = \sum_n L_n$ in a power series in ϵ and its transpose (more accurately, L_n is a sum of monomials of degree n, where each of the monomials is a product of ϵ and its transpose). Using this expansion, (13) can be rewritten as two recursion relations

$$\begin{aligned}L_{n+1} - L_{n+1}^{\mathrm{T}} &= L_n^{\mathrm{T}}\eta^{-1}\epsilon^{\mathrm{T}} - \epsilon\eta^{-1}L_n\,,\\ L_{n+1} + L_{n+1}^{\mathrm{T}} &= -\sum_{k=1}^{n} L_k\eta^{-1}L_{n-k+1}^{\mathrm{T}}\,,\end{aligned} \quad \text{(C.2)}$$

which, altogether, determine L order by order in the expansion. From this condition, and from taking $L = \eta$ when the change of coordinate is trivial ($\epsilon = 0$), we find the recursive solution

$$\begin{aligned}L_0 &= \eta\,,\\ L_{n+1} &= \frac{1}{2}\left(L_n^{\mathrm{T}}\eta^{-1}\epsilon^{\mathrm{T}} - \epsilon\,\eta^{-1}L_n - \sum_{k=1}^{n} L_k\eta^{-1}L_{n-k+1}^{\mathrm{T}}\right)\,,\end{aligned} \quad \text{(C.3)}$$

where the last term in parentheses must be evaluated only for $n>1$. This determines L uniquely; up to cubic order, the explicit solution reads

$$L=\eta-\frac{1}{2}\left(\epsilon-\epsilon^{\mathrm{T}}\right)+\frac{1}{8}\left(3\epsilon^2-\epsilon\epsilon^{\mathrm{T}}-\epsilon^{\mathrm{T}}\epsilon-(\epsilon^T)^2\right) +\frac{1}{16}\left(\epsilon^2\epsilon^{\mathrm{T}}+\epsilon\,\epsilon^{\mathrm{T}}\epsilon+\epsilon(\epsilon^T)^2+\epsilon^{\mathrm{T}}\epsilon^2-\epsilon^{\mathrm{T}}\epsilon\,\epsilon^{\mathrm{T}} +(\epsilon^T)^2\epsilon+(\epsilon^T)^3-5\epsilon^3\right)+\dots \tag{C.4}$$

In this expression the presence of η^{-1} between any two consecutive ϵ or ϵ^{T} is understood. We can now use this information to determine the matrix $\Pi=\eta+\Pi_1+\Pi_2+\Pi_3+\dots$, see (27), where the subscript indicates the order in which the graviton f (before Weyl rescaling) and the pion field π appear in this expression. In fact, in this work we only need the expansion up to third order: we must evaluate the interaction term (26) up to quartic order in the fields, since we are at most interested in four-point interaction vertices. However, since one of the two factors entering in (26) does not have a background part, it is sufficient to expand Π at cubic order in the fluctuations.

From the definition (27), we see that we need to invert L up to cubic order. The inversion up to cubic order in ϵ is obtained in a straightforward manner from (C.4). However, one has then to realize that ϵ itself is an expansion series in terms of the fields that are contained in f and π. Therefore, we must now expand ϵ at cubic order in the physical fields. This is done expanding the definition of ϵ, (C.1), where (in matrix notation) $e=\eta+f$, while π enters in the inverse of $\partial x/\partial y$ as written in (28). The expansion up to cubic order for ϵ reads

$$\epsilon=f-\pi-\pi\eta^{-1}(f-\pi)+\pi\eta^{-1}\pi\eta^{-1}(f-\pi)+\dots \tag{C.5}$$

and, finally, L^{-1} is found to be

$$L^{-1}=\eta^{-1}+\frac{1}{2}\left(\pi^{\mathrm{T}}-\pi\right)+\frac{1}{8}\left(3\pi^2-(\pi^{\mathrm{T}})^2-\pi\pi^{\mathrm{T}}-\pi^{\mathrm{T}}\pi\right) +\frac{1}{4}\left(f\pi^{\mathrm{T}}+f\pi-\pi f-\pi^{\mathrm{T}}f\right) +\frac{1}{16}\left((\pi^{\mathrm{T}})^3-5\pi^3+\pi(\pi^{\mathrm{T}})^2+\pi^2\pi^{\mathrm{T}}+\pi\pi^{\mathrm{T}}\pi -\pi^{\mathrm{T}}\pi\pi^{\mathrm{T}}+\pi^{\mathrm{T}}\pi^2+(\pi^{\mathrm{T}})^2\pi\right) +\frac{1}{8}\left(2\pi^2f-f(\pi^{\mathrm{T}})^2-\pi f\pi+\pi^{\mathrm{T}}f\pi^{\mathrm{T}}-\pi f\pi^{\mathrm{T}} -f\pi^{\mathrm{T}}\pi+\pi^{\mathrm{T}}f\pi-f\pi^2+f\pi\pi^{\mathrm{T}}\right) +\frac{1}{8}\left(\pi f^2-f^2\pi^{\mathrm{T}}-f^2\pi+\pi^{\mathrm{T}}f^2\right)+\dots. \tag{C.6}$$

In both this expression and the next ones we have suppressed writing the matrix η^{-1} between consecutive factors. Inserting this expression in (27), we obtain

$$\Pi=\eta+\frac{1}{2}\left(\pi+\pi^{\mathrm{T}}\right)+\frac{1}{8}\left(3\pi\pi^{\mathrm{T}}-\pi^2-\pi^{\mathrm{T}}\pi-(\pi^{\mathrm{T}})^2\right) +\frac{1}{4}\left(f\pi^{\mathrm{T}}+f\pi-\pi f-\pi^{\mathrm{T}}f\right) +\frac{1}{16}\left(\pi^3+(\pi^{\mathrm{T}})^3-\pi(\pi^{\mathrm{T}})^2-\pi^2\pi^{\mathrm{T}}-\pi\pi^{\mathrm{T}}\pi -\pi^{\mathrm{T}}\pi\pi^{\mathrm{T}}+\pi^{\mathrm{T}}\pi^2+(\pi^{\mathrm{T}})^2\pi\right) +\frac{1}{8}\left(\pi f\pi-f(\pi^{\mathrm{T}})^2+\pi^{\mathrm{T}}f\pi^{\mathrm{T}}+\pi f\pi^{\mathrm{T}}-f\pi^{\mathrm{T}}\pi +\pi^{\mathrm{T}}f\pi-f\pi^2+f\pi\pi^{\mathrm{T}}-2\pi\pi^{\mathrm{T}}f\right) +\frac{1}{8}\left(\pi f^2-f^2\pi^{\mathrm{T}}-f^2\pi+\pi^{\mathrm{T}}f^2\right)+\dots, \tag{C.7}$$

which we finally insert in the interaction term (26). This gives the interactions between the various polarizations of the graviton for the model we are discussing.

Equation (39) of the main text included the dominant interaction terms up to quartic order. These leading terms are obtained from the general expressions just given, by substituting (28) in the expression above, and only keeping terms with at most a single f or two A's. (All other terms have fewer derivatives, and so do not control the high energy limit of the model.) This gives

$$\Pi=\eta+\Phi+\frac{1}{2}\left(A+A^{\mathrm{T}}\right)+\frac{1}{4}\Phi(F-2f) -\frac{1}{4}(F-2f)\Phi+\frac{1}{8}\left(3A\,A^{\mathrm{T}}-A^2-(A^{\mathrm{T}})^2-A^{\mathrm{T}}A\right) +\frac{1}{8}\left(F\Phi^2-\Phi^2F+FA^{\mathrm{T}}\Phi-\Phi A\,F+A^{\mathrm{T}}\Phi A-A\Phi A^{\mathrm{T}} +4\Phi f\Phi-2f\Phi^2-2\Phi^2f\right). \tag{C.8}$$

Notice that many possible structures are absent. In particular all terms with higher powers of Φ and no other fields have canceled. The reason for this is that in (C.7) terms that could give such terms have coefficients that add up to zero. For example, substituting $\pi\to\Phi$ in the combination $(3\pi\pi^{\mathrm{T}}-\pi^2-(\pi^{\mathrm{T}})^2-\pi^{\mathrm{T}}\pi)$ gives zero because Φ is symmetric. This completes the development of the expansions of the various functions that appear in the main text to the order required there.

Appendix D: Exact expressions

In addition to the perturbative expansions presented in the previous appendix, it is also possible to derive closed exact expressions. Such results can be obtained as follows: by multiplying the relations

$$\frac{\partial x}{\partial y}e\eta^{-1}=e'L^{-1}\,,\quad \eta^{-1}e\left(\frac{\partial x}{\partial y}\right)^{\mathrm{T}}=(L^{\mathrm{T}})^{-1}e'\,, \tag{D.1}$$

we can obtain an equation for $\eta^{-1}e'$ of which we can take the formal square root,

$$e'=\eta\left[\eta^{-1}\frac{\partial x}{\partial y}e\eta^{-1}e\left(\frac{\partial x}{\partial y}\right)^{\mathrm{T}}\right]^{1/2}. \tag{D.2}$$

Substituting this back into one of the equations (D.1), we find

$$L = e\left(\frac{\partial x}{\partial y}\right)^{\mathrm{T}}\left[\eta^{-1}\frac{\partial x}{\partial y}e\eta^{-1}e\left(\frac{\partial x}{\partial y}\right)^{\mathrm{T}}\right]^{-1/2}. \quad \text{(D.3)}$$

This in turn results in

$$\Pi = \frac{\partial y}{\partial x}\left[\frac{\partial x}{\partial y}e\,\eta^{-1}e\left(\frac{\partial x}{\partial y}\right)^{\mathrm{T}}\eta^{-1}\right]^{-1/2}\frac{\partial x}{\partial y}e\,. \quad \text{(D.4)}$$

References

1. A.G. Riess et al., Astron. J. **116**, 1009 (1998)
2. S. Perlmutter et al., Astrophys. J. **517**, 565 (1999)
3. D.N. Spergel et al., Astrophys. J. Suppl. **148**, 175 (2003)
4. T. Damour, I.I. Kogan, A. Papazoglou, Phys. Rev. D **66**, 104 025 (2002)
5. N. Arkani-Hamed, H.C. Cheng, M.A. Luty, S. Mukohyama, JHEP **0405**, 074 (2004)
6. G.R. Dvali, G. Gabadadze, M. Porrati, Phys. Lett. B **485**, 208 (2000)
7. C. Deffayet, G.R. Dvali, G. Gabadadze, Phys. Rev. D **65**, 044 023 (2002)
8. M.A. Luty, M. Porrati, R. Rattazzi, JHEP **0309**, 029 (2003)
9. A. Nicolis, R. Rattazzi, JHEP **0406**, 059 (2004)
10. C. Charmousis, R. Gregory, N. Kaloper, A. Padilla, arXiv:hep-th/0604086
11. M. Fierz, W. Pauli, Proc. R. Soc. London A **173**, 211 (1939)
12. Particle Data Group, S. Eidelman et al., Phys. Lett. B **592**, 1 (2004)
13. V. Rubakov, arXiv:hep-th/0407104
14. S.L. Dubovsky, JHEP **0410**, 076 (2004)
15. B.M. Gripaios, JHEP **0410**, 069 (2004)
16. N. Arkani-Hamed, H. Georgi, M.D. Schwartz, Ann. Phys. **305**, 96 (2003)
17. C.J. Isham, A. Salam, J.A. Strathdee, Phys. Rev. D **3**, 867 (1971)
18. S. Groot Nibbelink, M. Peloso, Class. Quantum Grav. **22**, 1313 (2005)
19. H. van Dam, M.J.G. Veltman, Nucl. Phys. B **22**, 397 (1970)
20. V.I. Zakharov, JETP Lett. **12**, 312 (1970)
21. A.I. Vainshtein, Phys. Lett. B **39**, 393 (1972)
22. C. Deffayet, G.R. Dvali, G. Gabadadze, A.I. Vainshtein, Phys. Rev. D **65**, 044 026 (2002)
23. D.G. Boulware, S. Deser, Phys. Rev. D **6**, 3368 (1972)
24. G. Gabadadze, A. Gruzinov, Phys. Rev. D **72**, 124 007 (2005)
25. P. Creminelli, A. Nicolis, M. Papucci, E. Trincherini, JHEP **0509**, 003 (2005)
26. C. Deffayet, J.W. Rombouts, Phys. Rev. D **72**, 044 003 (2005)
27. M.D. Schwartz, Phys. Rev. D **68**, 024 029 (2003)